高等代数 (II) 第八次习题课

李卓远　　数学科学学院

zy.li@stu.pku.edu.cn

1　内容概要

- 内积空间, 酉空间.
- 标准正交基和 Schmidt 正交化.
- 斜 Hermite 变换.

2　补充知识

2.1　内积空间与酉空间

设 $V \in \mathbf{Vect}_{\mathbb{F}}$, $f : V \times V \to \mathbb{F}$. 内积空间和酉空间有如下联系.

	内积空间	酉空间
$\mathbb{F} =$	$\mathbb{R}$	$\mathbb{C}$
symmetric / Hermite	$(\alpha, \beta) = (\beta, \alpha)$	$(\alpha, \beta) = \overline{(\beta, \alpha)}$
positive-definite	$(\alpha, \alpha) \geq 0$, $(\alpha, \alpha) = 0 \iff \alpha = 0$	
linearity	$(k_1\alpha_1 + k_2\alpha_2, \beta) = k_1(\alpha_1, \beta) + k_2(\alpha_2, \beta)$	
length	$\lVert\alpha\rVert = \sqrt{(\alpha, \alpha)}$	
Cauchy-Bunyakovsky-Schwarz	$\lvert(\alpha, \beta)\rvert \leq \lVert\alpha\rVert \lVert\beta\rVert$	
angle between 2 vectors	$\langle\alpha, \beta\rangle = \arccos \frac{(\alpha,\beta)}{\lVert\alpha\rVert\lVert\beta\rVert}$	
orthogonal	$\alpha \perp \beta \iff (\alpha, \beta) = 0$	
parallelogram law	$\lVert\alpha + \beta\rVert^2 + \lVert\alpha - \beta\rVert^2 = 2\lVert\alpha\rVert^2 + 2\lVert\beta\rVert^2$	
Pythagorean theorem	$\Re(\alpha, \beta) = 0 \iff \lVert\alpha + \beta\rVert^2 = \lVert\alpha\rVert^2 + \lVert\beta\rVert^2$	
polarization identities	$(\alpha, \beta) = \frac{1}{4}\sum_{k=0}^{1}(-1)^k\lVert\alpha + (-1)^k\beta\rVert^2$	$(\alpha, \beta) = \frac{1}{4}\sum_{k=0}^{3}\mathrm{i}^k\lVert\alpha + \mathrm{i}^k\beta\rVert^2$
度量矩阵	实对称正定	Hermite 正定
标准正交基组成的矩阵	正交矩阵	酉矩阵
相似标准形	正交相似于实对角阵	酉相似于实对角阵

Proposition 2.1.1 (inner product, norm, metric and topology)**.** *内积 (inner product), 范数 (norm), 度量 (metric) 和拓扑 (topology) 有如下关系:*

- *Each inner product space induces a canonical norm* $\lVert\alpha\rVert = \sqrt{(\alpha, \alpha)}$*;*

- *Each normed space induces a distance (induced metric)* $d(\alpha,\beta)=\|\alpha-\beta\|$*;*
- **Each metric space induces a topology with a collection of basis* $\{B(\alpha,\varepsilon)\}_{\varepsilon>0,\alpha}$.

Example 2.1.2. *内积空间和赋范空间的例子:*

- ℓ^p*: normed space, not inner product space for any* $p\neq 2$ *(check the paralleogram law).*
  - ℓ^1*: absolutely convergent series;*
  - ℓ^2*: square summable series;*
  - ℓ^∞*: bounded sequence.*
- $L^2[-\pi,\pi]$ *(complex valued). One orthonormal basis:* $\{\mathrm{e}^{\mathrm{i}nx}\}_{n\in\mathbb{Z}}$.

$$(f,g):=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\overline{g(x)}\mathrm{d}x.$$

Proposition 2.1.3 (Legendre polynomials)**.** 设

$$P_k=\frac{1}{2^k k!}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left((x^2-1)^k\right).$$

定义 $\mathbb{R}[x]$ 上的内积

$$(f,g)=\int_{-1}^{1}f(x)g(x)\mathrm{d}x.$$

则 $\{P_k\}_{k=0}^{\infty}$ 构成一组正交基.

证明. 显然 $\deg P_k=2k-k=k$, 故而 $\{P_k\}_{k=0}^{\infty}$ 线性无关, 下面只需证明其正交性, 事实上只需验证

$$\int_{-1}^{1}P_k(x)x^l\mathrm{d}x=0,\ \forall l<k.$$

由于

$$\begin{aligned}\int_{-1}^{1}P_k(x)x^l\mathrm{d}x&=\frac{1}{2^k k!}\int_{-1}^{1}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left((x^2-1)^k\right)x^l\mathrm{d}x\\&=\frac{1}{2^k k!}\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}\left((x^2-1)^k\right)x^l\Big|_{-1}^{1}-\frac{l}{2^k k!}\int_{-1}^{1}\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}\left((x^2-1)^k\right)x^{l-1}\mathrm{d}x,\end{aligned}$$

其中

$$\begin{aligned}\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}\left((x^2-1)^k\right)&=\sum_{p=0}^{k-1}\binom{k-1}{p}\frac{\mathrm{d}^p}{\mathrm{d}x^p}\left((x-1)^k\right)\frac{\mathrm{d}^{k-1-p}}{\mathrm{d}x^{k-1-p}}\left((x+1)^k\right)\\&=\sum_{p=0}^{k-1}\binom{k-1}{p}\frac{k!}{(k-p)!}(x-1)^{k-p}\frac{k!}{(p+1)!}(x+1)^{p+1}\end{aligned}$$

当 $x=\pm1$ 时均为 0, 故而

$$\begin{aligned}\int_{-1}^{1}P_k(x)x^l\mathrm{d}x&=-\frac{l}{2^k k!}\int_{-1}^{1}\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}}\left((x^2-1)^k\right)x^{l-1}\mathrm{d}x\\&=\cdots=\frac{(-1)^l l!}{2^k k!}\int_{-1}^{1}\frac{\mathrm{d}^{k-l}}{\mathrm{d}x^{k-l}}\left((x^2-1)^k\right)\mathrm{d}x=0.\end{aligned}$$

□

Remark 2.1.4 (Chebyshev polynomials)**.** *若考虑* $\mathbb{R}[x]$ *上另一内积*

$$(f,g) := \int_{-1}^{1} f(x)g(x)(1-x^2)^{1/2}\mathrm{d}x,$$

相应的正交基可选为 $T_k(\cos\theta) = \cos k\theta$, $k = 0,1,2,\cdots$. $\deg T_k = k$.

2.2 Schmidt 正交化

已知 $\dim V = n$ 及一组基 $\alpha_1,\alpha_2,\cdots,\alpha_n$, 求一组标准正交基. 注意到标准正交基与正交基只相差一个数乘, 故不妨先求得一组正交基再进行归一化处理. 设

$$\begin{aligned}
&\beta_1 = \alpha_1;\\
&\beta_2 = \alpha_2 - \frac{(\alpha_2,\beta_1)}{(\beta_1,\beta_1)}\beta_1;\\
&\beta_3 = \alpha_3 - \frac{(\alpha_3,\beta_1)}{(\beta_1,\beta_1)}\beta_1 - \frac{(\alpha_3,\beta_2)}{(\beta_2,\beta_2)}\beta_2;\\
&\cdots\cdots\\
&\beta_n = \alpha_n - \sum_{k=1}^{n-1}\frac{(\alpha_n,\beta_k)}{(\beta_k,\beta_k)}\beta_k.
\end{aligned}$$

再将 $\beta_1,\beta_2,\cdots,\beta_n$ 归一化即得一组标准正交基 $\hat\beta_1\hat\beta_2,\cdots,\hat\beta_n$. 不难看出对于上述操作总有

$$\beta_k \in \alpha_k + \mathrm{span}(\beta_1,\cdots,\beta_{k-1}) = \alpha_k + \mathrm{span}(\alpha_1,\cdots,\alpha_{k-1}),$$

于是

$$(\beta_1,\cdots,\beta_k,\cdots,\beta_n) = (\alpha_1,\cdots,\alpha_k,\cdots,\alpha_n)\begin{pmatrix}1 & & * & & *\\ & \ddots & & \ddots & \\ & & 1 & & *\\ & & & \ddots & \\ 0 & & & & 1\end{pmatrix}.$$

特别地, 考虑 $V = \mathbb{R}^n$, 则上述变换实际上给出了矩阵 $(\alpha_1,\cdots,\alpha_n)$ 的 QR 分解.

$$\begin{aligned}
(\alpha_1,\cdots,\alpha_k,\cdots,\alpha_n) &= (\beta_1,\cdots,\beta_k,\cdots,\beta_n)\begin{pmatrix}1 & & * & & *\\ & \ddots & & \ddots & \\ & & 1 & & *\\ & & & \ddots & \\ 0 & & & & 1\end{pmatrix}^{-1}\\
&= (\hat\beta_1,\cdots,\hat\beta_k,\cdots,\hat\beta_n)D\begin{pmatrix}1 & & * & & *\\ & \ddots & & \ddots & \\ & & 1 & & *\\ & & & \ddots & \\ 0 & & & & 1\end{pmatrix}^{-1} =: QR,
\end{aligned}$$

其中 Q 为正交矩阵, R 为上三角矩阵.

3 典型例题

Problem 3.1. 设 V 为内积空间, $\mathscr{H}:\alpha\mapsto\alpha-2(\alpha,w)w,\ w\in V$.

- 判断 w 满足什么条件时 $\mathscr{H}$ 为正交变换.
- 若 $V=\mathbb{R}^m$ 内积定义为 $(\alpha,\beta)=\alpha^\mathsf{T}\beta$, 给出 $\mathscr{H}$ 的矩阵形式. 证明此时对任意 $\beta\in\mathbb{R}^n$ 都存在适当的 w 使得
$$H\beta=\|\beta\|e_1.$$

证明. 设 $\alpha,\beta\in V$, 则

$$(\mathscr{H}\alpha,\mathscr{H}\beta)=(\alpha-2(\alpha,w)w,\beta-2(\beta,w)w)=(\alpha,\beta)-4(1-(w,w))(\alpha,w)(\beta,w).$$

即 $\mathscr{H}$ 为正交变换当且仅当 w 满足 $(w,w)=1$ 或 $w=0$. 若考虑 $V=\mathbb{R}^m$, 那么 $\mathscr{H}$ 所对应的矩阵形式为 $H=I-2ww^\mathsf{T}$.

$$H\beta=\|\beta\|e_1 \iff \beta-2(w^\mathsf{T}\beta)w=\|\beta\|e_1 \implies w/\!/(\beta-\|\beta\|e_1).$$

若 $\beta=0$, 取 $w=0$ 即可, 否则取 $w=\frac{\beta-\|\beta\|e_1}{\|\beta-\|\beta\|e_1\|}$, 代入验证可知

$$\begin{aligned}
2w^\mathsf{T}\beta&=2\|\beta-\|\beta\|e_1\|^{-1}(\beta-\|\beta\|e_1)^\mathsf{T}\beta\\
&=\|\beta-\|\beta\|e_1\|^{-1}(2\|\beta\|^2-2\|\beta\|e_1^\mathsf{T}\beta)\\
&=\|\beta-\|\beta\|e_1\|^{-1}(\|\beta\|^2-2\|\beta\|e_1^\mathsf{T}\beta+\|\|\beta\|e_1\|^2)\\
&=\|\beta-\|\beta\|e_1\|.
\end{aligned}$$

□

Problem 3.2. 设 V 为有限维线性空间, f 为 V 上的非退化对称/斜对称双线性函数, 子空间 $W_1\subseteq W_2\subseteq V$. 求证 $W_1^\perp/W_2^\perp\cong(W_2/W_1)^*$.

证明. 构造

$$\begin{aligned}
\varphi: W_1^\perp/W_2^\perp&\to(W_2/W_1)^*\\
\alpha+W_2^\perp&\mapsto\langle(\beta+W_1)\mapsto f(\alpha,\beta)\rangle.
\end{aligned}$$

首先验证其良定义性. 设 $\alpha_1+W_2^\perp=\alpha_2+W_2^\perp$, $\beta_1+W_1=\beta_2+W_1$, 则 $\alpha_1-\alpha_2\in W_2^\perp$, $\beta_1-\beta_2\in W_1$,

$$\begin{aligned}
\varphi(\alpha_1+W_1)(\beta_1+W_1)&=f(\alpha_1,\beta_1)\\
&=f(\alpha_1,\beta_2)+f(\alpha_1,\beta_1-\beta_2)\\
&=f(\alpha_2,\beta_2)+f(\alpha_1-\alpha_2,\beta_2)+f(\alpha_1,\beta_1-\beta_2)=f(\alpha_2,\beta_2).
\end{aligned}$$

再验证 φ 为线性映射 (略去细节). 下面说明同构, 注意到 V 为有限维空间, 比较维数可知只需说明 φ 为单射, 这由 f 非退化是显然的. □

Problem 3.3. 设 $\eta_1,\cdots,\eta_5$ 为内积空间 V 的一组标准正交基, 令

$$\alpha_1=\eta_1+2\eta_3-\eta_5;\quad \alpha_2=\eta_2-\eta_3+\eta_4;\quad \alpha_3=-\eta_2+\eta_3+\eta_5.$$

求 $\operatorname{span}(\alpha_1,\alpha_2,\alpha_3)$ 的一组标准正交基.

证明. 由题意可知

$$(\alpha_1,\alpha_2,\alpha_3)=(\eta_1,\cdots,\eta_5)\begin{pmatrix}1&&\\&1&-1\\2&-1&1\\&1&\\-1&&1\end{pmatrix}.$$

直接进行 Schmidt 正交化:

$$\begin{aligned}
\beta_1&=\alpha_1\\
\beta_2&=\alpha_2-\frac{(\alpha_2,\beta_1)}{(\beta_1,\beta_1)}\beta_1=\alpha_2+\frac{1}{3}\alpha_1;\\
\beta_3&=\alpha_3-\frac{(\alpha_3,\beta_1)}{(\beta_1,\beta_1)}\beta_1-\frac{(\alpha_3,\beta_2)}{(\beta_2,\beta_2)}\beta_2=\alpha_3-\frac{1}{6}\beta_1-\frac{5}{7}\beta_2.
\end{aligned}$$

于是

$$(\beta_1,\beta_2,\beta_3)=(\alpha_1,\alpha_2,\alpha_3)\begin{pmatrix}1&1/3&1/14\\0&1&5/7\\0&0&1\end{pmatrix}=(\eta_1,\cdots,\eta_5)\begin{pmatrix}1&1/3&1/14\\&1&-2/7\\2&-1/3&3/7\\&1&5/7\\-1&-1/3&13/14\end{pmatrix}.$$

验算可知其列正交. 最终将 β_1, β_2, β_3 归一化, 得

$$\hat{\beta}_1=\frac{1}{\sqrt{6}}\beta_1,\quad \hat{\beta}_2=\sqrt{\frac{3}{7}}\beta_2,\quad \hat{\beta}_3=\sqrt{\frac{14}{23}}\beta_3.$$

□

Problem 3.4 (正交变换和斜对称变换)**.** *设 V 为内积空间. Recall:*

- *正交变换: $(A\alpha,A\beta)=(\alpha,\beta)$, $\forall\alpha,\beta$;*
- *斜对称变换: $(A\alpha,\beta)+(\alpha,A\beta)=0$, $\forall\alpha,\beta$.*

证明:

a) 若 A 斜对称, $(A\pm I)$ 可逆, 则 $B=(A\pm I)(A\mp I)^{-1}$ 正交;

b) 若 B 正交, $(B\pm I)$ 可逆, 则 $A=(B\mp I)(B\pm I)^{-1}$ 斜对称.

证明. *a*) 任取 $\alpha\in V$. 设 $\beta=(A\mp I)^{-1}\alpha$, 则有 $A\beta\mp\beta=\alpha$, 于是

$$\begin{aligned}
(B\alpha,B\alpha)&=((A\pm I)\beta,(A\pm I)\beta)=(A\beta,A\beta)+(\beta,\beta),\\
(\alpha,\alpha)&=(A\beta\mp\beta,A\beta\mp\beta)=(A\beta,A\beta)+(\beta,\beta).
\end{aligned}$$

故 $(B\alpha,B\alpha)=(\alpha,\alpha)$, 即 B 为保距变换. 再证保距变换均为正交变换.

$$\|\alpha+\beta\|^2=\|B(\alpha+\beta)\|^2=(B\alpha,B\alpha)+2(B\alpha,B\beta)+(B\beta,B\beta)=\|\alpha\|^2+2(B\alpha,B\beta)+\|\beta\|^2,$$

这蕴含着 $(B\alpha,B\beta)=(\alpha,\beta)$, 于是 B 为正交变换.

b) 任取 $\alpha \in V$, 令 $\beta = (B \pm I)^{-1}\alpha$, 则有 $\alpha = B\beta \pm \beta$, 于是

$$(A\alpha, \alpha) = ((B \mp I)\beta, (B \pm I)\beta) = (B\beta, B\beta) - (\beta, \beta) = 0.$$

那么有

$$0 = (A(\alpha+\beta), \alpha+\beta) = (A\alpha, \alpha) + (A\alpha, \beta) + (A\beta, \alpha) + (A\beta, \beta) = (A\alpha, \beta) + (\alpha, A\beta).$$

故 A 为斜对称变换.

$\square$

Remark 3.5. *酉矩阵和 Hermite 矩阵也有类似关系:*

- *H 为 Hermite 矩阵, 则 $(I \pm \mathrm{i}H)$ 可逆, 且 $U = (I \mp \mathrm{i}H)(I \pm \mathrm{i}H)^{-1}$ 为酉矩阵;*
- *U 为酉矩阵且 $(I \pm \mathrm{i}U)$ 可逆, 则 $H = \mp\mathrm{i}(I \mp \mathrm{i}U)(I \pm \mathrm{i}U)^{-1}$ 为 Hermite 矩阵.*

Problem 3.6. *设 $\mathscr{A}$ 为酉空间 V 上的变换, 满足*

$$(\mathscr{A}\alpha, \beta) + (\alpha, \mathscr{A}\beta) = 0, \forall \alpha, \beta \in V.$$

称这样的变换 $\mathscr{A}$ 为斜 Hermite 变换. 证明

(1) V 上的斜 Hermite 变换为线性变换, 且线性变换 $\mathscr{A}$ 为斜 Hermite 变换当且仅当其在 V 的一个标准正交基下的矩阵表示 A 为斜 Hermite 矩阵: $A^* = -A$.

(2) 斜 Hermite 变换的特征值实部总为 0.

证明. 对任意 $\alpha, \beta \in V$,

$$(\mathscr{A}(k\alpha, l\beta), \gamma) = -(k\alpha + l\beta, \mathscr{A}\gamma) = -k(\alpha, \mathscr{A}\gamma) - l(\beta, \mathscr{A}\gamma) = (k\mathscr{A}\alpha + l\mathscr{A}\beta, \gamma).$$

故 $\mathscr{A}$ 为线性变换. 设 $\{\alpha_1, \cdots, \alpha_n\}$ 为 V 的一个标准正交基,

$$\mathscr{A}(\alpha_1, \cdots, \alpha_n) = (\alpha_1, \cdots, \alpha_n)A,$$

则 $\mathscr{A}$ 为斜 Hermite 变换当且仅当 $(\mathscr{A}\eta_p, \eta_q) = -(\eta_p, \mathscr{A}\eta_q) = -\overline{(\mathscr{A}\eta_q, \eta_p)}, \forall p, q$. 即 $A^* = -A$. 在这一条件下, 设 λ 为 $\mathscr{A}$ 的特征值, η 为对应的一个特征向量, 则

$$\lambda(\eta, \eta) = (\mathscr{A}\eta, \eta) = -(\eta, \mathscr{A}\eta) = -(\eta, \lambda\eta) = -\overline{\lambda}(\eta, \eta),$$

利用 $\eta \neq 0$ 可知 $\lambda + \overline{\lambda} = 0$, 即 λ 的实部为 0. $\square$