Advanced algebra (II) supplemental materials # 1

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1 Preliminary

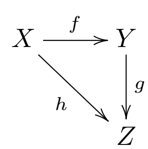
Definition 1.1. *A category $\mathscr{C}$ consists of a collection of objects X, Y, Z, ... (e.g., sets, vector spaces) and a collection of morphisms between them $f : X \to Y$, $g : Y \to Z$, ... (e.g., maps, linear maps resp.) with some additional constraints.*

Definition 1.2. *The collection of morphisms from X to Y is defined as $\mathrm{Hom}(X, Y)$. Two objects X and Y are isomorphic if there exists $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(Y, X)$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$ are both "identities".*

Example 1.3.
- **Set** *is the category consisting of all the sets and maps between them;*
- $\mathbf{Vect}_k$ *is the category consisting of all the vector spaces over field k and linear maps between them.*

*Note that under such definitions, the collection of all the objects of a category does not necessarily form a set.

Definition 1.4. *Within a certain category $\mathscr{C}$, the following triangular diagram*

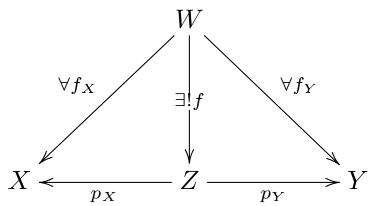


is said to commute if $g \circ f = h$. Also, an analogous definition holds for square diagrams.

2 universal property and duality

2.1 products and coproducts

Definition 2.1 (binary products)**.** *Let X and Y be objects of a category $\mathscr{C}$ (denoted by X, $Y \in \mathscr{C}$). A binary product of X and Y (if it exists) is defined as an object equipped with morphisms $p_X : Z \to X$ and $p_Y : Z \to Y$ such that for any morphisms $f_X : W \to X$ and $f_Y : W \to Y$, there* *exists a unique morphism* $f : W \to Z$ *satisfying* $p_X \circ f = f_X$, $p_Y \circ f = f_Y$.



Proposition 2.2. *In the category* **Set***, binary products always exist. More precisely,* $X \times Y$ *together with the two coordinate projections* p_X *and* p_Y *is a binary product of* X *and* Y.

Proof. The existence of f follows immediately by defining

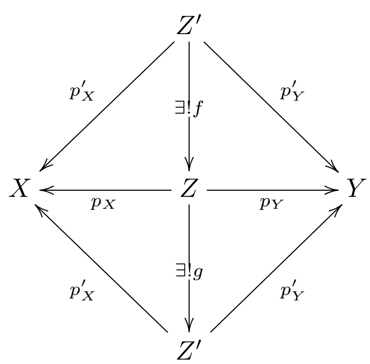
$$f : W \to X \times Y,\ w \mapsto (f_X(w), f_Y(w)).$$

Clearly such f is unique to ensure that $p_X \circ f = f_X$ and $p_Y \circ f = f_Y$. □

Exercise 2.3. *By imitating the proof of Proposition 2.2, show that* $X \oplus Y$ *equipped with two linear maps is a product of* X *and* Y *in* $\mathbf{Vect}_k$.

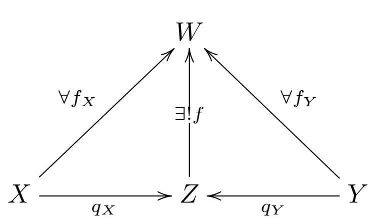
Proposition 2.4. *The binary product of two objects* X *and* Y *is unique up to isomorphism if it exists. In other words, if* Z *equipped with* p_X, p_Y *and* Z' *equipped with* p'_X, p'_Y *are both binary product of* X *and* Y*, then* Z *and* Z' *are isomorphic.*

Proof. The proposition immediately follows by



and the uniqueness of f and g gives $g \circ f = 1_{Z'}$, $f \circ g = 1_Z$. □

Definition 2.5 (binary coproducts)**.** *Let* X *and* Y *be objects of a category* $\mathscr{C}$ *(denoted by* X, $Y \in \mathscr{C}$*). A binary coproduct of* X *and* Y *(if it exists) is defined as an object equipped with morphisms* $q_X : X \to Z$ *and* $q_Y : Y \to Z$ *such that for any morphisms* $f_X : X \to W$ *and* $f_Y : Y \to W$*, there exists a unique morphism* $f : Z \to W$ *satisfying* $f \circ q_X = f_X$, $f \circ q_Y = f_Y$.



Exercise 2.6. *Try to define a coproduct of two sets X and Y in* **Set**.

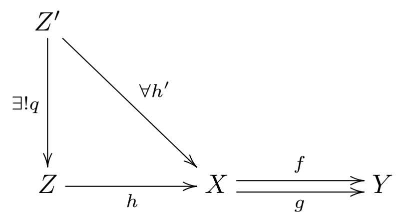
Exercise 2.7. *By imitating the proof of Proposition 2.2, show that $X \oplus Y$ equipped with two linear maps is also a coproduct of X and Y in* $\mathbf{Vect}_k$.

Exercise 2.8. *By imitating the proof of Proposition 2.4, show that the binary coproduct of two objects X and Y is unique up to isomorphism.*

2.2 kernels and cokernels

Definition 2.9. *Given f, $g \in \mathrm{Hom}(X, Y)$, an object Z equipped with a morphism $h \in \mathrm{Hom}(Z, X)$ is an equalizer if*

- $f \circ h = g \circ h$;
- *for any $h' \in \mathrm{Hom}(Z', X)$ satisfying $f \circ h' = g \circ h'$ there exists a unique morphism $q \in \mathrm{Hom}(Z', Z)$ satisfying $h \circ q = h'$.*



Exercise 2.10. *The equalizer of f, $g \in \mathrm{Hom}(X, Y)$ is unique (if exists) up to isomorphism, which is denoted by* $\mathrm{eq}(f, g)$.

Exercise 2.11. *For f, $g \in \mathrm{Hom}(X, Y)$ in* **Set**, *the subset of X where f and g coincides equipped with the canonical embedding map is an equalizer.*

Exercise 2.12. *For any $f \in \mathrm{Hom}(X, Y)$ in $\mathbf{Vect}_k$, show that $\ker(f) = \mathrm{eq}(f, 0)$, where 0 stands for the zero map.*

Exercise 2.13. *Can you find the relations between the two definitions of products and coproducts? Try to give a "proper" definition of "cokernels" and then show their uniqueness up to isomorphism.*

Exercise 2.14. *Denote by $\mathrm{coeq}(f, g)$ the coequalizer of f, $g \in \mathrm{Hom}(X, Y)$. Show that $\mathrm{coker}(f) = \mathrm{coeq}(f, 0)$ in $\mathbf{Vect}_k$.*

Remark 2.15. *See `https://ncatlab.org/nlab/show/HomePage` for details on all the definitions mentioned above.*