Advanced algebra (II) supplemental materials # 2

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We always assume that $V \in \mathbf{Vect}_{\mathbb{F}}$ with $\operatorname{char} \mathbb{F} = 0$ in the following discussion.

1 Commutators and Lie algebras

Definition 1.1. *Given any $\mathscr{A}, \mathscr{B} \in \operatorname{End}(V)$, the commutator of $\mathscr{A}$ and $\mathscr{B}$ is defined as*

$$[\mathscr{A}, \mathscr{B}] := \mathscr{A}\mathscr{B} - \mathscr{B}\mathscr{A}.$$

Definition 1.2. *A Lie algebra is a vector space L together with a binary operation $[-,-] : L \times L \to L$ satisfying*

1. *(bilinearity) $[x, -]$, $[-, x] \in \operatorname{End}(L)$ for any $x \in L$;*
2. *(alternativity) $[x, x] = 0$ for any $x \in L$;*
3. *(Jacobi identity) For any $x, y, z \in L$,*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Exercise 1.3. *Check that the vector space $\operatorname{End}(V)$ together with the commutator operation is a Lie algebra. From now on, we only discuss such kind of Lie algebra.*

Definition 1.4. *For the Lie algebra $(\operatorname{End}(V), [-,-])$, we can define the adjoint mapping*

$$\operatorname{ad}_{\mathscr{A}} = [\mathscr{A}, -] \in \operatorname{End}(\operatorname{End}(V))$$

for each $\mathscr{A} \in \operatorname{End}(V)$. i.e.,

$$\operatorname{ad}_{\mathscr{A}}(\mathscr{B}) = [\mathscr{A}, \mathscr{B}] = \mathscr{A}\mathscr{B} - \mathscr{B}\mathscr{A} \in \operatorname{End}(V)$$

for any $\mathscr{A}, \mathscr{B} \in \operatorname{End}(V)$.

Exercise 1.5. *Check that for any $\mathscr{A}, \mathscr{B}, \mathscr{X} \in \operatorname{End}(V)$,*

$$\operatorname{ad}_{\mathscr{X}}(\mathscr{A}\mathscr{B}) = \operatorname{ad}_{\mathscr{X}}(\mathscr{A})\mathscr{B} + \mathscr{A} \operatorname{ad}_{\mathscr{X}}(\mathscr{B}).$$

Try to find the connections with the derivative defined in calculus. Moreover, show that

$$\operatorname{ad}_{\mathscr{X}}(\mathscr{A}^k) = k\mathscr{A}^{k-1} \operatorname{ad}_{\mathscr{X}}(\mathscr{A})$$

if $\mathscr{A}$ and $\operatorname{ad}_{\mathscr{X}}(\mathscr{A})$ commute.

2 Triangularization

In this section, we assume that $\dim V = n$.

Definition 2.1. *A collection $(A_j)_j \subseteq \mathrm{End}(V)$ (not necessarily finite) can be simultaneously triangularized if one of the following holds.*

- *There exists an invertible P s.t. $P^{-1}A_jP$ is an upper triangular matrix for each j.*
- *There exist some subspace $W_k \subseteq V$, $k = 1, 2, \cdots, n$ satisfying*

  a) $\dim W_k = k$, $k = 1, 2, \cdots, n$;

  b) $0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{n-1} \subseteq W_n = V$;

  c) W_k *is* A_j*-invariant for any* k *and* j.

Exercise 2.2. *Prove: the two conditions above are equivalent, and they also imply that $(A_j)_j$ have a common eigenvector.*

Exercise 2.3. *For $A \in \mathrm{End}(V)$ and an A-invariant subspace W of V,*

$$A/W : V/W \to V/W,\ u + W \mapsto Au + W$$

is well-defined. Abusing of notation, we still use $A = A/W \in \mathrm{End}(V/W)$.

Theorem 2.4. *Let the collection $(A_j)_j$ satisfy $\mathscr{P}$ on every invariant subspace of V, where $\mathscr{P}$ stands for certain property. If*

- *for any $(A_j)_j$-invariant subspace M and N with $N \subseteq M$, $(A_j)_j$ satisfies $\mathscr{P}$ on M/N;*
- *for any W with $\dim W > 1$, $(A_j)_j$ satisfies $\mathscr{P}$ on W implies that W contains a non-trivial $(A_j)_j$-invariant subspace,*

then $(A_j)_j$ can be simultaneously triangularized.

Proof. Choose a maximal chain of invariant subspaces:

$$0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_{m-1} \subseteq W_m = V.$$

Here, "maximal" means that $W_{k-1} \neq W_k$ and that there does not exist any other invariant subspace W' strictly lying between W_{k-1} and W_k. The existence can be guaranteed since V only has finite dimension. If $\dim(W_k/W_{k-1}) > 1$ for some $k = 1, 2, \cdots, m$, then W_k/W_{k-1} has a non-trivial invariant subspace $Z + W_{k-1}$ by the assumptions above. It follows that Z is an invariant subspace lying strictly between W_{k-1} and W_k, contradiction. Hence $\dim W_k = \dim W_{k-1} + 1$ for each k. ☐

Corollary 2.5 ($\mathscr{P}$ ="pair-wise commutative")**.** *Let $(A_j)_j$ be commutative operators on* $\mathrm{End}(V)$*. $\mathbb{F} = \mathbb{C}$. Show that they can be simultaneously triangularized.*

Proof. It suffices to show that for any W, $\dim W > 1$ implies that W has a non-trivial invariant subspace. If each A_j is a multiple of the identity, then any 1-d subspace of W will be invariant. Otherwise we assume without loss of generality A_1 is not a multiple of the identity. Since $\mathbb{F} = \mathbb{C}$, A_1 has an eigenspace Z of dimension less than $\dim W$. By the commutativity, such Z is a non-trivial A_j-invariant subspace for each j. ☐

Exercise 2.6 ($\mathscr{P}$ ="$\mathrm{rank}(AB - BA) \leq 1$")**.** *Let* $\mathrm{rank}(AB - BA) \leq 1$ *for* $A, B \in \mathrm{End}(V)$*. Show that they can be simultaneously triangularized. (Hint: one of* $\ker(B - \lambda I)$ *and* $\mathrm{Im}(B - \lambda I)$ *is A-invariant for eigenvalue λ of B)*

3 Some more exercises

In this section, we assume that $\dim V = n$, $\mathbb{F} = \mathbb{C}$ and $A, B, X \in \mathrm{End}(V)$. Recall that

$$\mathrm{ad}_A \in \mathrm{End}(\mathrm{End}(V)),\ \mathrm{ad}_A(X) = AX - XA.$$

Let X be an eigenvector of ad_A corresponding with eigenvalue λ.

Exercise 3.1. *Show that*

1. *A can be simultaneously triangularized with X.*
2. *$s \neq 0$ implies that X is nilpotent. Find a counterexample for $s = 0$.*

Exercise 3.2. *If A is nilpotent, then $s = 0$.*

Exercise 3.3. $\mathrm{ad}_A(\mathrm{ad}_A(B)) = 0$ *implies that* $\mathrm{ad}_A(B)$ *is nilpotent.*